The oscillating plate problem in magnetohydrodynamics

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The oscillating plate problem is investigated for the case of an incompressible, electrically conducting fluid in the presence of a magnetic field. The boundary conditions are examined in detail, and a solution is found with the aid of suitable approximations. The motion of the fluid is shown to consist mainly of magnetohydrodynamic waves, but there is also a viscous boundary layer in which the solution agrees with that given by other writers.

1. Introduction

Certain difficulties occur in the theory of magnetohydrodynamic boundary layers which may possibly be resolved by an examination of the corresponding linear problem in which the fluid motion is caused by an infinite flat plate moving in its own plane. Ludford (1959) has considered the Rayleigh problem (impulsive motion of the flat plate) including magnetohydrodynamic effects, and in the present note the related case is examined in which the plate executes simple harmonic motion and the flow is quasi-steady. There is an advantage in considering this case, since the mathematical difficulties encountered by Ludford are then avoided.

The case in which the kinematic viscosity is small compared with the magnetic diffusivity is examined in detail, since this is likely to be the situation in experimental work. The most interesting effect found is the production of magnetohydrodynamic waves which become diffused due to the finite electrical conductivity of the fluid. A viscous boundary layer occurs at the surface of the plate, and the solution in this region agrees exactly with that found by Ong & Nicholls (1959), using an approximate equation given by Rossow (1957). The reasons for this agreement and also the approximations involved in Ong & Nicholls' solution are discussed.

The implications of the assumption made by Ludford, that the boundary consists of material having infinite electrical conductivity, are examined in detail, and necessary conditions for its validity are given.

2. Equations and boundary conditions

The equations* describing plane laminar motion of an incompressible conducting fluid in the absence of a pressure gradient are

$$\left(\eta \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right) H + H_0 \frac{\partial u}{\partial y} = 0, \tag{1}$$

* M.K.S. units are used throughout. Since Ludford has given a neat form of the solution of the algebraic equation (11), his notation is used for convenience.

$$A_0^2 \frac{\partial H}{\partial y} + H_0\left(\nu \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial t}\right) u = 0, \tag{2}$$

where the motion is in the x-direction, and the velocity $\mathbf{v}$, magnetic field $\mathbf{H}$, and current $\mathbf{j}$ are given by

$$\mathbf{v} = (u, 0, 0), \quad \mathbf{H} = (H, H_0, 0), \quad \mathbf{j} = (0, 0, j). \tag{3}$$

H_0 is a constant magnetic field perpendicular to the plane of motion, which is used to define an Alfvén velocity as follows:

$$A_0^2 = \mu H_0^2/\rho. \tag{4}$$

There are two diffusivities involved in the equations, namely the kinematic viscosity ν, and the magnetic diffusivity, η, which is defined as

$$\eta = 1/\mu\sigma, \tag{5}$$

where μ is the permeability and σ the conductivity of the fluid. The equation for the pressure is

$$\frac{\partial}{\partial y}\left(p + \frac{\mu H^2}{2}\right) = 0, \tag{6}$$

and the current is given by

$$j = -\frac{\partial H}{\partial y}. \tag{7}$$

We shall consider a quasi-steady motion, neglecting transients, in which the time-dependent part of the solution is a factor $e^{i\omega t}$. Hence the operator $\partial/\partial t$ in equations (1) and (2) can be replaced by $i\omega$, and it is then easily shown that the general solution of the equations is

$$u = [A\, e^{my} + B\, e^{-my} + C\, e^{ny} + D\, e^{-ny}]\, e^{i\omega t}, \tag{8}$$

$$H = \frac{H_0}{A_0^2}\left[\left(\frac{i\omega}{m} - m\nu\right) A\, e^{my} + \ldots\right] e^{i\omega t}, \tag{9}$$

$$j = -\frac{H_0}{A_0^2}[(i\omega - \nu m^2)\, A\, e^{my} + \ldots]\, e^{i\omega t}, \tag{10}$$

where A, B, C and D are arbitrary constants and $r = \pm m, \pm n$, are the roots of the quartic

$$(\eta r^2 - i\omega)(\nu r^2 - i\omega) - A_0^2 r^2 = 0. \tag{11}$$

Ludford has shown that

$$m = (a + ib\omega)^{\frac{1}{2}} + (a + ic\omega)^{\frac{1}{2}}, \tag{12}$$

$$n = (a + ib\omega)^{\frac{1}{2}} - (a + ic\omega)^{\frac{1}{2}}, \tag{13}$$

where

$$a = \frac{A_0^2}{4\eta\nu}, \quad b = (\eta^{\frac{1}{2}} + \nu^{\frac{1}{2}})^2/4\eta\nu, \quad c = (\eta^{\frac{1}{2}} - \nu^{\frac{1}{2}})^2/4\eta\nu. \tag{14}$$

Under laboratory conditions it is usually found that $\eta \gg \nu$*, and in these circumstances, b and c are very nearly equal. Hence we can write

$$b = d(1+\epsilon), \quad c = d(1-\epsilon),$$

where

$$d = 1/4\nu, \quad \epsilon = 2(\nu/\eta)^{\frac{1}{2}},$$

and so

$$(a + ib\omega)^{\frac{1}{2}} = (a + id\omega)^{\frac{1}{2}}\left\{1 + \frac{i\omega d\epsilon}{2(a + id\omega)} + O(\epsilon^2)\right\},$$

$$(a + ic\omega)^{\frac{1}{2}} = (a + id\omega)^{\frac{1}{2}}\left\{1 - \frac{i\omega d\epsilon}{2(a + id\omega)} + O(\epsilon^2)\right\}.$$

* For instance, $\nu/\eta \approx 10^{-7}$ for mercury.

Substituting from (14), and neglecting all but the terms of lowest order in ϵ, equations (12) and (13) give

$$m \approx (A_0^2 + i\omega\eta)^{\frac{1}{2}}/\sqrt{(\eta\nu)}, \tag{15}$$

$$n \approx i\omega/(A_0^2 + i\omega\eta)^{\frac{1}{2}}. \tag{16}$$

Thus, in the general solution, the terms with exponent $(i\omega t \pm my)$ contain the combination $y/\nu^{\frac{1}{2}}$, showing that there is an ordinary viscous boundary layer outside of which the fluid is affected only by the oscillations of the magnetic field. The terms with exponent $(i\omega t \pm ny)$ are independent of ν and represent diffused Alfvén waves moving along the lines of force of the undisturbed field $(0, H_0, 0)$.

The simplest example of laminar periodic motion is that caused by a solid boundary at $y = 0$ which oscillates in the x-direction with velocity $Ue^{i\omega t}$. The fluid is assumed to extend to infinity in $y > 0$, and the space $y < 0$ is occupied by solid material of uniform composition. We shall use primes to denote quantities in the solid region, which contains a magnetic field of the form $(H', H_0', 0)$, where H_0' is constant and H' is an induced field satisfying the equation

$$\frac{\partial H'}{\partial t} = \eta' \frac{\partial^2 H'}{\partial y^2}. \tag{17}$$

The appropriate solution of this equation which vanishes for large negative values of y is

$$H' = F \exp\left[\left(\frac{\omega}{2\eta'}\right)^{\frac{1}{2}} y + i\left\{\omega t + \left(\frac{\omega}{2\eta'}\right)^{\frac{1}{2}} y\right\}\right], \tag{18}$$

where F is a further arbitrary constant.

The disturbance must vanish as y tends to infinity, and so A and C must be zero since the real parts of m and n are positive. We are therefore required to determine three constants, namely B, D and F, from the conditions at the boundary $y = 0$.

The first condition is that there should be no slip at the boundary, that is

$$u = u' = Ue^{i\omega t} \quad \text{on} \quad y = 0. \tag{19}$$

There are two conditions on the magnetic field, requiring that the normal component of $\mu\mathbf{H}$ and the tangential component of $\mathbf{H}$ should be continuous. Hence

$$\mu H_0 = \mu' H_0', \tag{20}$$

and

$$H = H' \quad \text{on} \quad y = 0. \tag{21}$$

The remaining condition is on the tangential component of the electric field, which must also be continuous. Thus

$$\frac{1}{\sigma}\frac{\partial H}{\partial y} + \mu u H_0 = \frac{1}{\sigma'}\frac{\partial H'}{\partial y} + \mu' u' H_0' \quad \text{on} \quad y = 0,$$

which reduces, on using (19) and (20), to

$$\frac{1}{\sigma}\frac{\partial H}{\partial y} = \frac{1}{\sigma'}\frac{\partial H'}{\partial y} \quad \text{on} \quad y = 0. \tag{22}$$

The three conditions (19), (21) and (22) are sufficient to determine the three constants B, D and F.

3. Simplifying approximations

A simplified form of these boundary conditions is possible under certain circumstances. First, it can be seen from (18) that the field in the solid is confined* to a layer near the surface with thickness of order $(\eta'/\omega)^{\frac{1}{2}}$. Similarly the field induced in the fluid is confined to a layer with thickness at least of order $(\eta/\omega)^{\frac{1}{2}}$. Thus if $\eta \gg \eta'$, the field induced in the solid can be neglected, and the current associated with this induced field can be assumed to form a current sheet on the boundary which allows a discontinuity in the field. In this case, F does not have to be calculated, and the two conditions required to find B and D are (19) and

$$\left(\frac{\partial H}{\partial y}\right)_{y=0} = \frac{\sigma}{\sigma'}\left(\frac{\omega}{2\eta'}\right)^{\frac{1}{2}} e^{i\pi/4} H(0). \tag{23}$$

The second possible simplification is found by considering the ratio $\left|\frac{L}{H}\frac{\partial H}{\partial y}\right|_{y=0}$, where L is some typical length. This is found to be of the order of $\left(\frac{\sigma\mu'}{\mu\sigma'}\right)^{\frac{1}{2}}$. Thus provided

$$\sigma/\sigma' \ll \mu/\mu', \tag{24}$$

the boundary condition (23) can be taken to be

$$\frac{\partial H}{\partial y} = 0 \quad \text{on} \quad y = 0, \tag{25}$$

which is the condition used by Ludford. In general μ and μ' will be approximately equal, unless the solid is of a ferromagnetic material, when μ' will be much larger than μ. Whatever the case, (24) is a sufficient condition for $\eta \gg \eta'$, and for the boundary conditions to be (19) and (25) with F zero. If (24) is not satisfied, but $\eta \gg \eta'$, then the boundary conditions are (19) and (23) with F zero. Finally, if η is not much greater than η', the boundary conditions must be (19), (22) and (23), and F is not zero.

We shall consider a case in which the condition (24) holds, and it is easily found from (19) and (25) that

$$B = (\nu n^2 - i\omega)\, U/\nu(n^2-m^2),$$
$$D = (i\omega - \nu m^2)\, U/\nu(n^2-m^2).$$

Hence

$$u = \frac{Ue^{i\omega t}}{\nu(n^2-m^2)}\left[(\nu n^2 - i\omega)\, e^{-my} - (\nu m^2 - i\omega)\, e^{-ny}\right], \tag{26}$$

$$H = UH_0 e^{i\omega t}\frac{(\nu n^2 - i\omega)(\nu m^2 - i\omega)}{A_0^2\nu(n^2-m^2)}\left[\frac{e^{-ny}}{n} - \frac{e^{-my}}{m}\right]. \tag{27}$$

When $\eta \gg \nu$, m and n are given by (15) and (16), and these solutions for u and H become

$$u = \frac{Ue^{i\omega t}}{\left(1+\frac{i\omega\eta}{A_0^2}\right)}\left[\exp\left\{-\frac{i\omega y}{(A_0^2+i\omega\eta)^{\frac{1}{2}}}\right\} + \frac{i\omega\eta}{A_0^2}\exp\left\{-\frac{y(A_0^2+i\omega\eta)^{\frac{1}{2}}}{\sqrt{(\eta\nu)}}\right\}\right], \tag{28}$$

$$H = \frac{UH_0 e^{i\omega t}}{A_0\left(1+\frac{i\omega\eta}{A_0^2}\right)^{\frac{3}{2}}}\left[\left(1+\frac{i\omega\eta}{A_0^2}\right)\exp\left\{-\frac{i\omega y}{(A_0^2+i\omega\eta)^{\frac{1}{2}}}\right\} - \frac{i\omega\sqrt{(\eta\nu)}}{A_0^2}\exp\left\{-\frac{y(A_0^2+i\omega\eta)^{\frac{1}{2}}}{\sqrt{(\eta\nu)}}\right\}\right]. \tag{29}$$

* The 'skin effect'.

The solutions in the viscous boundary layer are found by taking the limit as y and ν tend to zero while $y/\nu^{\frac{1}{2}}$ remains finite. These boundary layer solutions are

$$u = \frac{Ue^{i\omega t}}{\left(1+\frac{i\omega\eta}{A_0^2}\right)}\left[1+\frac{i\omega\eta}{A_0^2}\exp\left\{-\frac{y(A_0^2+i\omega\eta)^{\frac{1}{2}}}{\sqrt{(\eta\nu)}}\right\}\right], \tag{30}$$

$$H = \frac{UH_0 e^{i\omega t}}{A_0\left(1+\frac{i\omega\eta}{A_0^2}\right)^{\frac{1}{2}}}. \tag{31}$$

This expression for u is exactly that found by Ong & Nicholls, using the equation

$$\frac{\partial u}{\partial t}+\frac{A_0^2}{\eta}(u-Ue^{i\omega t}) = \nu\frac{\partial^2 u}{\partial y^2}, \tag{32}$$

first given by Rossow. This confirms the result found by Ludford, that Rossow's equation gives the correct solution in the viscous boundary layer provided $\eta \gg \nu$. The truth of this can be found from an examination of equation (1), for

$$\frac{\partial H}{\partial t} \ll \eta\frac{\partial^2 H}{\partial y^2},$$

if

$$UL \ll \eta, \tag{33}$$

where L is a length typical of the region under consideration. For the viscous boundary layer, L is ν/U, and (33) is simply $\eta \gg \nu$. Neglecting the term $\frac{\partial H}{\partial t}$, equation (1) can be integrated to give

$$\eta\frac{\partial H}{dy} = H_0(Ue^{i\omega t}-u), \tag{34}$$

using the boundary condition (25), and substitution of this result into equation (2) yields (32). The chief drawbacks of Rossow's equation are that it cannot be used to give the solution for H, nor does it give any indication of the effect of the magnetic field on the motion of the fluid outside the viscous boundary layer. Because of this, Rossow's approach will fail in many cases in which the flow outside the viscous boundary layer is affected by the magnetic field, as it gives no indication of the correct condition at the outer edge of the boundary layer. There may also be some doubt about the validity of using (25) as a boundary condition in some circumstances.

The solution for the flow and field outside the boundary layer are found from (28) and (29) by allowing ν to tend to zero whilst y remains finite, giving

$$u = \frac{U}{\left(1+\frac{i\omega\eta}{A_0^2}\right)}\exp\left\{i\omega t-\frac{i\omega y}{(A_0^2+i\omega\eta)^{\frac{1}{2}}}\right\}, \tag{35}$$

$$H = \frac{UH_0}{A_0\left(1+\frac{i\omega\eta}{A_0^2}\right)^{\frac{1}{2}}}\exp\left\{i\omega t-\frac{i\omega y}{(A_0^2+i\omega\eta)^{\frac{1}{2}}}\right\} = \frac{H_0}{A_0}\left(1+\frac{i\omega\eta}{A_0^2}\right)^{\frac{1}{2}}u. \tag{36}$$

Thus the disturbance is a diffused Alfvén wave propagating in the direction of increasing y. This can be seen more readily if a low frequency ($\omega \ll A_0^2/\eta$) approximation is made, for the above equations then become

$$u = U \exp\left\{-\frac{\omega^2\eta}{2A_0^3}y + i\omega\left(t - \frac{y}{A_0}\right)\right\}, \tag{37}$$

$$H = \frac{UH_0}{A_0}\exp\left\{-\frac{\omega^2\eta}{2A_0^3}y + i\omega\left(t - \frac{y}{A_0}\right)\right\}, \tag{38}$$

showing that the fluid and field move together, and the disturbance is an Alfvén wave which is damped out over a distance of order $2A_0^3/\omega^2\eta$. This is the result obtained by Alfvén (1950, p. 82) in an investigation of the effect of finite conductivity of the fluid on plane magnetohydrodynamic waves.

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